On the conditional symmetries of Levi and Winternitz

1990 J. Phys. A: Math. Gen. 23 3643

(http://iopscience.iop.org/0305-4470/23/15/033)

J. Phys. A: Math. Gen. 23 (1990) 3643-3645. Printed in the UK

COMMENT

On the conditional symmetries of Levi and Winternitz

Giuseppe Gaeta†

Institut des Hautes Etudes Scientifiques, 91440 Bures sur Yvette, France, and Centre de Physique Theorique, Ecole Polytechnique, 91128 Palaiseau, France

Received 9 April 1990

Abstract. It is pointed out that the 'conditional symmetries' of Levi and Winternitz, and the 'non-classical method' first introduced by Bluman and Cole, have a natural group theoretical setting. This corresponds to classifying solutions to a PDEs according to their full symmetry and not just according to their symmetries which are also symmetries of the PDE.

Recently, Levi and Winternitz [1] pointed out the relevance of the 'non-classical method', introduced by Bluman and Cole [2] (see also [3, 4]), in the quest of solutions to PDEs by means of reductions to an ODE [5, 6].

We would like to shortly look at their procedure, which introduces the useful concept of *conditional symmetries*, in a way which we think clarifies it further. We assume the reader to know of [1] and to be familiar with the general concepts of symmetry of differential equations and their solutions, see e.g. [5] to which we conform also for notation.

Given a differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{1}$$

with base space M,

$$M = X \times U \tag{2}$$

where $X \subseteq \mathbf{R}^q$ is the space of independent variables and $U \subseteq \mathbf{R}^p$ that of dependent ones, and $u^{(n)}$ the nth prolongation of u, so that

$$\Delta : M^{(n)} \to \mathbf{R}^s \tag{3}$$

($s = 1$ for a scalar equation; $s > 1$ for a vector one, i.e. a system of scalar equations), we can look for its symmetry algebra $\mathscr{G}_\Delta$, with nth prolongation $\mathscr{G}_\Delta^{(n)}$.

This is the algebra of vector fields on M, which we write as

$$v = \sum_{i=1}^{q} \xi^i \partial_{x^i} + \sum_{h=1}^{p} \phi^h \partial_{u^h} \tag{4}$$

or shortly

$$v = \xi\partial_x + \phi\partial_u \tag{5}$$

† CNR fellow, grant 203.01.48.

such that the solution manifold S_Δ of Δ,

$$S_\Delta = \{(x, u^{(n)}) \in M^{(n)} / \Delta(x, u^{(n)}) = 0\} \subset M^{(n)} \tag{6}$$

is invariant under them, or properly speaking under their nth prolongation:

$$\mathscr{G}_\Delta = \{v \in \mathrm{Diff}(M) / v^{(n)} : S_\Delta \to TS_\Delta\} \subset \mathrm{Diff}(M). \tag{7}$$

This condition is also written

$$(v^{(n)} \cdot \Delta)|_{S_\Delta} = 0 \tag{8}$$

but we will keep to definition (7); this implicitly identifies, for symmetry purposes, a differential equation (1) and its solution manifold (6).

A vector field $v \in \mathrm{Diff}(M)$ also induces an action on functions $f: X \to U$ (see e.g. [5, section 2.2]), with infinitesimal action

$$(I + \varepsilon v) : f^h(x) \to \tilde{f}^h_\varepsilon(\tilde{x}_\varepsilon) = f^h(x) + \varepsilon\left(\phi^h - \xi^i \frac{\partial f^h}{\partial x^i}\right) + \mathrm{O}(\varepsilon^2) \tag{9}$$

and generator, with obvious notation

$$v_{(f)} = \phi^h \partial_{f^h} - \xi^i \partial_{x^i}. \tag{10}$$

We can define the symmetry (or isotropy) algebra of a function $f: X \to U$ as

$$\mathscr{G}_f = \{v \in \mathrm{Diff}(M) / v_{(f)} \cdot f = 0\} \subset \mathrm{Diff}(M) \tag{11}$$

(obviously $v \in \mathscr{G}_f$ also implies $v_f^{(n)} \cdot f^{(n)} = 0$, with $f^{(n)}$ the nth prolongation of f).

If we denote $\mathscr{G} \equiv \mathrm{Diff}(M)$ and

$$\mathscr{G}_f^\Delta = \mathscr{G}_\Delta \cap \mathscr{G}_f \tag{12}$$

we have the subgroups diagram

$$\begin{array}{ccc} & \mathscr{G} & \\ \swarrow & & \searrow \\ \mathscr{G}_\Delta & & \mathscr{G}_f \\ \searrow & & \swarrow \\ & \mathscr{G}_f^\Delta & \end{array} \tag{13}$$

Now, the classical reduction method consists in this: given $\mathscr{G}_\Delta$, we consider its subalgebras $\mathscr{G}_i \subset \mathscr{G}_\Delta$, and for each of these we make the ansatz $\mathscr{G}_i \subseteq \mathscr{G}_f^\Delta$; with this the equation (1) reduces to a simpler one, possibly an ODE.

Actually, one does not consider all the (in general, infinitely many) $\mathscr{G}_i$, but a representative for each stratum [7, 8] in the stratification of $\mathscr{S}_\Delta$ by $\mathscr{G}_\Delta^{(n)}$, the prolongation of $\mathscr{G}_\Delta$ (see e.g. [5, section 3.3], where this enters through the concept of an 'optimal system' of subgroups), where

$$\mathscr{S}_\Delta = \{f: X \to U / \Delta(x, f^{(n)}) = 0\} \tag{14}$$

i.e. $\mathscr{S}_\Delta$ is the (functional) space of solutions to (1).

Once such a solution $u = f(x)$; $\mathscr{G}_i \subseteq \mathscr{G}_f$ is determined, symmetry copies of it (i.e. distinct solutions which are symmetry related to it) are obtained as

$$f_g(x) = g \cdot f(x) \qquad \forall_g \in \mathrm{G}_\Delta \tag{15}$$

where G_Δ is the connected Lie group generated by $\mathscr{G}_\Delta$; actually in (15) it suffices to consider

$$g \in \mathrm{G}_\Delta / \mathrm{G}_f^\Delta \tag{16}$$

as $g \in G_f^{\Delta}$ (the group generated by $\mathcal{G}_f^{\Delta}$) acts on f as the identity; symmetry-related solutions will be in one-to-one correspondence with elements of G_{Δ}/G_f^{Δ}.

Now, it is clear that we could as well stratify $\mathcal{S}_{\Delta}$ by considering $\mathcal{G}_f$ *tout court* and not just $\mathcal{G}_f^{\Delta}$; this means that we can classify solutions to $\Delta = 0$ according to their full symmetry under subgroups of $\mathcal{G} \equiv \mathrm{Diff}(M)$ and not just under subgroups of $\mathcal{G}_{\Delta} \subset \mathcal{G}$.

The reduction procedure, once $\mathcal{G}_f$ is chosen, is then as before: the ansatz $\mathcal{G}_f = \mathcal{G}_i \subset \mathcal{G}$ partially determines the form of f (its graph $\Gamma_f = (x, f(x))$ must be the union of $\mathcal{G}_f$ invariants in M), and we are left with an equation simpler than (1), possibly an ODE, to solve.

Notice that, since $\mathcal{G}_f^{\Delta} \subseteq \mathcal{G}_f$, this reduction is equally or more effective than the one based on $\mathcal{G}_f^{\Delta}$; on the other hand, in order to obtain in this way all the solutions with a given G_f^{Δ}, we should consider (representatives of the strata for) all the $\mathcal{G}_f$ such that $\mathcal{G}_f^{\Delta} = \mathcal{G}_f \cap \mathcal{G}_{\Delta}$.

The classical reduction method corresponds to classification of solutions (stratification of $\mathcal{S}_{\Delta}$) according to $\mathcal{G}_f^{\Delta}$; the non-classical reduction method corresponds to classification (stratification) according to $\mathcal{G}_f$.

It is clear that the 'conditional symmetries' correspond to generators v in $\mathcal{G}_f^{\mathrm{cond}}$,

$$\mathcal{G}_f^{\mathrm{cond}} \equiv \mathcal{G}_f / \mathcal{G}_f^{\Delta}. \tag{17}$$

It is also obvious that they do not in general form an algebra; this is in fact the case if and only if $\mathcal{G}_f^{\Delta}$ is an ideal in $\mathcal{G}_f$,

$$[\mathcal{G}_f, \mathcal{G}_f^{\Delta}] \subseteq \mathcal{G}_f^{\Delta} \tag{18}$$

or equivalently if G_f^{Δ} is a normal subgroup of G_f,

$$G_f^{\Delta} \lhd G_f \tag{19}$$

while in general cases $\mathcal{G}_f^{\mathrm{cond}}$ is a coset space.

Notice that, in view of (10), the condition to have $v \in \mathcal{G}_f$, v given by (5), can just be written as a differential equation in $M^{(1)}$, i.e.

$$\Delta_{(s)}(x, u^{(1)}) \equiv \xi u_x - \phi = 0 \tag{20}$$

which is the 'side condition', or auxiliary equation (see (2.4) of [1]), of Levi and Winternitz.

References

[1] Levi D and Winternitz P 1989 Non-classical symmetry reduction: example of the Boussinesq equation *J. Phys. A: Math. Gen.* **22** 2915
[2] Bluman G W and Cole J D 1969 The general similarity solution of the heat equation *J. Math. Mech.* **18** 1025
[3] Olver P J and Rosenau P 1986 The construction of special solutions to partial differential equations *Phys. Lett.* **114A** 107
[4] Olver P J and Rosenau P 1987 Group invariant solutions of differential equations *SIAM J. Appl. Math.* **47** 263
[5] Olver P J 1986 *Application of Lie Groups to Differential Equations* (Berlin: Springer)
[6] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[7] Michel L 1972 Nonlinear group action: smooth actions of compact Lie groups on manifolds *Statistical Mechanics and Field Theory* (Jerusalem: Israel University Press)
[8] Abud M and Sartori G 1983 The geometry of spontaneous symmetry breaking *Ann. Phys.* **150** 307